STOCHASTIC COMPARISONS OF EXTREMES FOR GENERALIZED WEIBULL AND BETA-WEIBULL DISTRIBUTIONS: SIMULATION AND APPLICATION TO COVID-19 IN BURKINA FASO

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Abstract

Comparing the intensity of epidemic waves across regions or time periods is a major public health decision-making issue. This article addresses this problem by developing a theoretical framework for stochastically comparing extremes (minimums and maximums) of random variables following generalized Weibull and Beta-Weibull distributions. The approach relies on vector majorization to rank parameter dispersion and on stochastic orders to compare peaks and troughs of epidemic waves. The established theorems are validated by numerical simulations. An application to daily COVID-19 data from Burkina Faso (13 regions, March 2020 – March 2022) is carried out. The results show that wave 2 (December 2020 – February 2021) was the most intense (mean $\theta = 10.92$) and that the Centre region (Ouagadougou) concentrates the highest peaks (θ reaching 77.9). The generalized Beta-Weibull model provides a better fit than the classical Weibull model, but its estimation is fragile (saturation of parameter a at 100). The comparison theorems are validated for the Weibull model (5 out of 6 pairs). The matrix chain majorization analysis (simultaneous comparison of θ and γ) shows that no wave satisfies the required conditions, which in itself is an epidemiological finding: the intensity and shape of regional peaks are independent. Limitations include the disparity in the number of adjusted regions across waves and the low activity of wave 1. The study opens perspectives for the analysis of other infectious diseases (malaria, meningitis) and for extension to other countries in the West African subregion.

Keywords: Stochastic comparisons, majorization, Weibull distribution, generalized Beta-Weibull distribution, COVID-19 Burkina Faso.

Classification: 60E05, 60G70, 62N05, 62P10

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1 Introduction

The COVID-19 pandemic has highlighted the need for statistical tools capable of modeling and comparing the evolution of the epidemic across different regions or time periods. The analysis of extremes, i.e., contamination peaks and troughs, is a major public health decision-making issue. In Burkina Faso, as in many countries, understanding the dynamics of successive waves and comparing their intensity across territories remains a methodological challenge.

Reliability models provide a suitable framework for this problem. In reliability theory, the lifetime of a series system corresponds to the minimum of the lifetimes of its components, while that of a parallel system corresponds to the maximum Shaked and Shanthikumar (2007). These concepts find a natural echo in epidemic analysis: the peak of a wave (maximum) and the trough between two waves (minimum) can be studied using order statistics. Stochastic comparisons of these extremes allow answering questions such as: has a given region experienced stochastically higher peaks than another? Is one period more severe than another?

The research problem motivating this study can be formulated as follows: given $n$epidemic regions whose contamination data follow generalized Weibull or Beta-Weibull distributions with heterogeneous parameters, can we establish conditions on these parameters that guarantee that the peaks (maxima) or troughs (minima) of one configuration are stochastically higher than those of another? This question is directly motivated by the need to objectively compare the intensity of COVID-19 waves across regions of Burkina Faso.

The main objective of this study is to develop a methodological framework for stochastically comparing the extremes of random variables following Weibull and exponentiated Weibull distributions. More specifically, we aim to:

- Establish theoretical conditions, based on majorization Marshall et al. (2011), guaranteeing certain stochastic orders Shaked and Shanthikumar (2007) between minima and maxima.
- Apply these results to daily COVID-19 data from Burkina Faso, in order to compare the intensity of waves across different periods.

The remainder of this article is organized as follows. Section 2 presents a review of the recent literature. Section 3 details the materials and methods (study area, stochastic orders, majorization, Weibull and Beta-Weibull distributions, order statistics). Section 4 states and proves the theoretical results. Section 5 presents the numerical simulations. Section 6 applies these results to the COVID-19 data from Burkina Faso. A discussion of limitations and perspectives concludes the article.

2 Literature review

This section reviews recent work on stochastic comparisons of extremes, which form the foundation of our study.

The recent literature has grown richer with work on extreme comparisons for various distributions. Bhakta et al. (2024) established stochastic comparison conditions for finite mixture models with components following a generalized Weibull family, using usual stochastic orders, hazard rate orders, and likelihood ratio orders, relying on majorization concepts. Furthermore, Panahi et al. (2024) developed results for series and parallel systems subject to random shocks, also using vector majorization. The application of these methods to real data, particularly COVID-19 data, has recently been the subject of studies, such as that of Alshanbari et al. (2022) who proposed a new Burr-Weibull distribution to analyze the pandemic. Recently, Al-Jabbar et al. (2025)extended these comparisons to a five-parameter generalized Weibull distribution, using vector and matrix majorization tools.

These works demonstrate the interest of majorization and stochastic orders for comparing heterogeneous systems. Our study fits into this continuity by adapting these tools to generalized Weibull and Beta-Weibull distributions, and by applying them to COVID-19 data from Burkina Faso.

3 Materials and methods

The objective of this section is to present all the mathematical concepts and tools necessary for carrying out our study. We first introduce the main stochastic orders used to compare random variables Shaked and Shanthikumar (2007). Next, we present the theory of majorization, which provides a formal framework for comparing the dispersion of parameter vectors Marshall et al. (2011). These two methodological pillars form the foundation of the extreme comparisons developed in the remainder of the article.
We also recall the definitions and properties of the Weibull distribution McCool (2012), widely used in reliability analysis and lifetime modeling. Special attention is given to the family of generalized Beta-Weibull distributions, which extend the classical Weibull distribution by introducing additional shape parameters, allowing greater flexibility in fitting real data Cordeiro and de Castro (2011). Basic notions related to order statistics, essential for the study of extremes such as the minimum (series system) and the maximum (parallel system), are also presented Balakrishnan and Zhao (2013).
The results presented in this section refer mainly to the reference works Shaked and Shanthikumar (2007); Marshall et al. (2011) as well as to recent work on generalized Weibull distributions Cordeiro and de Castro (2011). The presentation of stochastic orders, majorization and order statistics follows the main lines of the approach developed by Al-Jabbar et al. (2025).

3.1 Study area

The study focuses on Burkina Faso, a West African country with an area of approximately 274,200 km². Administratively, the country had 13 regions at the time of data collection (according to the former administrative division): Boucle du Mouhoun, Cascades, Centre, Centre-Est, Centre-Nord, Centre-Ouest, Centre-Sud, Est, Haut-Bassins, Nord, Plateau-Central, Sahel and Sud-Ouest. The population of Burkina Faso is estimated at about 22 million inhabitants, with a high concentration in the Centre region (Ouagadougou, the capital).
Within the framework of the COVID-19 pandemic, these 13 regions were subject to daily epidemiological monitoring by the health authorities. The data used in this study cover the period from March 9, 2020 to March 18, 2022. They were collected by the national surveillance system and are accessible via the HERA (Health Emergency Response Africa) platform. Figure 3 illustrates the daily evolution of confirmed cases for the whole country.

3.2 Stochastic orders

Stochastic orders allow the comparison of two random variables according to different criteria Shaked and Shanthikumar (2007). Let X and Y be two positive random variables, with distribution functions F and G respectively, and survival functions $\bar{F} = 1 - F$ and $\bar{G} = 1 - G$. Let f and g denote their densities, $h_X = f/\bar{F}$ and $h_Y = g/\bar{G}$ their hazard rates, and $r_X = f/F$ and $r_Y = g/G$ their reversed hazard rates.
Usual stochastic order:X is said to be smaller than Y in the usual stochastic order, denoted $X \leq_{st} Y$, if $\bar{F}(t) \leq \bar{G}(t)$ for all tBelzunce et al. (2016). This means that Y tends to take larger values than X.
Hazard rate order:$X \leq_{hr} Y$ if $\bar{G}(t)/\bar{F}(t)$ is increasing in t, or equivalently if $h_X(t) \geq h_Y(t)$ for all tShaked and Shanthikumar (2007). This order indicates that Y fails less quickly than X.
Reversed hazard rate order:$X \leq_{rh} Y$ if $G(t)/F(t)$ is increasing in t, or equivalently if $r_X(t) \leq r_Y(t)$ for all tLi and Li (2013). This order is particularly useful for comparing maxima of random variables.
Likelihood ratio order:$X \leq_{lr} Y$ if $g(t)/f(t)$ is increasing in tDenuit and Mesfioui (2019). This is the strongest order: it implies the hazard rate and reversed hazard rate orders, which themselves imply the usual stochastic order Müller and Stoyan (2002).
The following implication relations are classical:

$$X \leq_{lr} Y \Rightarrow X \leq_{hr} Y \Rightarrow X \leq_{st} Y \quad \text{and} \quad X \leq_{lr} Y \Rightarrow X \leq_{rh} Y \Rightarrow X \leq_{st} Y.$$

3.3 Majorization

Majorization is a mathematical tool for comparing the dispersion of two vectors of real numbers Marshall et al. (2011). It is widely used in statistics to establish inequalities and stochastic comparisons.
Let $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{y} = (y_1, \dots, y_n)$ be two vectors in $\mathbb{R}^n$. Denote by $x_{(1)} \le \cdots \le x_{(n)}$ and $y_{(1)} \le \cdots \le y_{(n)}$ the components sorted in increasing order.
Majorization: The vector $\mathbf{x}$ is said to be majorized by $\mathbf{y}$, denoted $\mathbf{x} \preccurlyeq_m \mathbf{y}$, if $\sum_{i=1}^{k} x_{(i)} \ge \sum_{i=1}^{k} y_{(i)}$, for $k = 1, \dots, n-1$ and $\sum_{i=1}^{n} x_{(i)} = \sum_{i=1}^{n} y_{(i)}$. Intuitively, $\mathbf{x}$ is more "egalitarian" (less dispersed) than $\mathbf{y}$.
Weak supermajorization: The vector $\mathbf{x}$ is said to be weakly supermajorized by $\mathbf{y}$, denoted $\mathbf{x} \preccurlyeq^{w} \mathbf{y}$, if $\sum_{i=1}^{k} x_{(i)} \ge \sum_{i=1}^{k} y_{(i)}$, for $k = 1, \dots, n$.
Weak submajorization: The vector $\mathbf{x}$ is said to be weakly submajorized by $\mathbf{y}$, denoted $\mathbf{x} \preccurlyeq_w \mathbf{y}$, if $\sum_{i=k}^{n} x_{(i)} \le \sum_{i=k}^{n} y_{(i)}$, for $k = 1, \dots, n$.
Schur-convexity and Schur-concavity: A symmetric function $\varphi: \mathbb{R}^n \to \mathbb{R}$ is said to be Schur-convex if $\mathbf{x} \preccurlyeq_m \mathbf{y}$ implies $\varphi(\mathbf{x}) \le \varphi(\mathbf{y})$. It is said to be Schur-concave if the inequality is reversed Marshall et al. (2011). These properties are essential for deriving stochastic comparisons from parameter majorization.
In this paper, we use majorization to compare parameter vectors (e.g., the α_i or θ_i of Weibull and generalized Beta-Weibull distributions) and thus establish stochastic orders between the extremes of the studied systems.

3.4 Weibull distribution

The Weibull distribution is one of the most widely used distributions in reliability and lifetime data analysis. It is also used in epidemiology to model the temporal evolution of an epidemic McCool (2012). Its flexibility allows it to represent constant, increasing or decreasing hazard rates.
Definition: A random variable X follows a Weibull distribution with shape parameter $\gamma > 0$ and scale parameter $\theta > 0$ if its probability density function (PDF) is given by:

$$f(x;\theta,\gamma) = \frac{\gamma}{\theta}\left(\frac{x}{\theta}\right)^{\gamma-1} e^{-(x/\theta)^{\gamma}}, \quad x \ge 0.$$

Cumulative distribution function: The cumulative distribution function (CDF) is:

$$F(x;\theta,\gamma) = 1 - e^{-(x/\theta)^{\gamma}}, \quad x \ge 0.$$

Survival function: The survival function (or reliability) is:

$$\bar{F}(x;\theta,\gamma) = 1 - F(x;\theta,\gamma) = e^{-(x/\theta)^{\gamma}}, \quad x \ge 0.$$

Hazard rate function: The hazard rate is defined by $h(x) = f(x)/\bar{F}(x)$. For the Weibull distribution:

$$h(x;\theta,\gamma) = \frac{\gamma}{\theta}\left(\frac{x}{\theta}\right)^{\gamma-1}, \quad x \ge 0.$$

This rate is:

- constant if $\gamma = 1$ (exponential distribution),
- decreasing if $\gamma < 1$,
- increasing if $\gamma > 1$.

Quantile function (inverse): The quantile function is essential for generating random samples using the inverse transform method. For a probability level $u \in [0,1]$, the quantile function $F^{-1}(u)$ is obtained by solving $F(x) = u$:

$$F(x) = 1 - e^{-(x/\theta)^{\gamma}} = u \;\Rightarrow\; e^{-(x/\theta)^{\gamma}} = 1 - u.$$

Taking logarithms:

$$-(x/\theta)^{\gamma} = \ln(1-u) \;\Rightarrow\; (x/\theta)^{\gamma} = -\ln(1-u).$$

Hence $x = \theta\left(-\ln(1-u)\right)^{1/\gamma}$. Thus, the quantile function is:

$$F^{-1}(u;\theta,\gamma) = \theta\left(-\ln(1-u)\right)^{1/\gamma}, \quad u \in [0,1].$$

To generate a simulation, draw u from a uniform distribution on $[0,1]$ and compute $x = F^{-1}(u)$.

3.5 Generalized Beta-Weibull distribution

The generalized Beta-Weibull distribution is a flexible extension of the classical Weibull distribution, obtained by applying the beta transformation to the Weibull distribution Cordeiro and de Castro (2011). This family adds two additional shape parameters, allowing it to fit a wide variety of data shapes, including those with bathtub-shaped or unimodal hazard functions, which cannot be reproduced with the simple Weibull distribution. This flexibility is particularly valuable for modeling complex phenomena such as successive epidemic waves of COVID-19.

Definition: A random variable X follows a generalized Beta-Weibull distribution with parameters $\theta > 0$ (scale), $\gamma > 0$ (Weibull shape), $a > 0$ and $b > 0$(additional beta shape parameters) if its probability density function (PDF) is given by:

$$f(x;\theta,\gamma,a,b) = \frac{\gamma}{\theta}\left(\frac{x}{\theta}\right)^{\gamma-1} e^{-(x/\theta)^\gamma} \cdot \frac{\left[1-e^{-(x/\theta)^\gamma}\right]^{a-1}}{B(a,b)} \cdot \left(1-\left(1-e^{-(x/\theta)^\gamma}\right)\right)^{b-1}, \quad x \geq 0,$$

or, simplifying the expression:

$$f(x;\theta,\gamma,a,b) = \frac{\gamma}{\theta}\left(\frac{x}{\theta}\right)^{\gamma-1} e^{-b(x/\theta)^\gamma} \frac{\left[1-e^{-(x/\theta)^\gamma}\right]^{a-1}}{B(a,b)}, \quad x \geq 0,$$

where $B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ is the complete beta function Cordeiro and de Castro (2011).

Cumulative distribution function: The cumulative distribution function (CDF) of the generalized Beta-Weibull distribution is expressed using the ratio of incomplete beta functions:

$$F(x;\theta,\gamma,a,b) = I_{1-e^{-(x/\theta)^\gamma}}(a,b) = \frac{1}{B(a,b)}\int_0^{1-e^{-(x/\theta)^\gamma}} t^{a-1}(1-t)^{b-1}\,dt, \quad x \geq 0,$$

where $I_y(a,b)$ denotes the regularized incomplete beta function Cordeiro and de Castro (2011).

Survival function: The survival function is $\bar{F}(x;\theta,\gamma,a,b) = 1 - F(x;\theta,\gamma,a,b)$, $x \geq 0$.

Hazard rate function: The hazard rate is defined by $h(x) = f(x)/\bar{F}(x)$. For the generalized Beta-Weibull distribution, this rate can take much more varied forms than for the standard Weibull distribution: it can be increasing, decreasing, constant (special cases), but also bathtub-shaped (decreasing then increasing) or unimodal (increasing then decreasing), depending on the values of the parameters a and bCordeiro and de Castro (2011). This property is essential for modeling phenomena such as an epidemic, where the rate of new cases first increases, then decreases after the peak.

Quantile function (inverse): The quantile function is essential for generating random samples using the inverse transform method. For a probability level $u \in [0,1]$, we have $F(x) = u \Rightarrow I_{1-e^{-(x/\theta)^\gamma}}(a,b) = u$. There is no simple explicit form for x. In practice, we first compute $q = I_u^{-1}(a,b)$ (the inverse of the regularized incomplete beta function), then set:

$$1-e^{-(x/\theta)^\gamma} = q \;\Rightarrow\; e^{-(x/\theta)^\gamma} = 1-q \;\Rightarrow\; (x/\theta)^\gamma = -\ln(1-q).$$

Hence $x = \theta\left(-\ln(1-q)\right)^{1/\gamma}$, with $q = I_u^{-1}(a,b)$. Thus, the quantile function is:

$$F^{-1}(u;\theta,\gamma,a,b) = \theta\left(-\ln\left(1-I_u^{-1}(a,b)\right)\right)^{1/\gamma}, \quad u \in [0,1].$$

In simulation software such as R or Python, the quantile function can be obtained using the inverse beta function (e.g., `qbeta` in R).

Note: This distribution contains the Weibull distribution as a special case. Indeed, when $a = 1$ and $b = 1$, we have $B(1,1) = 1$ and the density reduces to $f(x;\theta,\gamma,1,1) = \frac{\gamma}{\theta}\left(\frac{x}{\theta}\right)^{\gamma-1} e^{-(x/\theta)^\gamma}$, which is exactly the density of the Weibull distribution with parameters θ (scale) and γ (shape). Thus, all results established for the Weibull distribution naturally extend to the generalized Beta-Weibull family Cordeiro and de Castro (2011).

3.6 Order statistics

Order statistics are a fundamental tool for studying extremes of a random sample. They are particularly useful in reliability for modeling the lifetime of series and parallel systems, as well as in epidemiology for analyzing peaks and troughs of an epidemic Balakrishnan and Zhao (2013).

Definition: Let $X_1, X_2, \ldots, X_n$ be independent random variables. Denote by $X_{1:n} \le X_{2:n} \le \cdots \le X_{n:n}$ the ordered values of this sample. The variable $X_{k:n}$ is called the k-th order statistic. The following special cases are at the heart of our study:

- $X_{1:n} = \min\{X_1, \ldots, X_n\}$: the minimum (series system),
- $X_{n:n} = \max\{X_1, \ldots, X_n\}$: the maximum (parallel system).

Cumulative distribution function of the maximum: For independent variables, the cumulative distribution function of the maximum is given by the product of the individual distribution functions:

$$F_{X_{n:n}}(x) = P(X_{n:n} \le x) = P(X_1 \le x, \ldots, X_n \le x) = \prod_{i=1}^{n} F_i(x).$$

This formula reflects the fact that the maximum is less than or equal to x if and only if all variables are.

Survival function of the minimum: The survival function of the minimum is obtained from the individual survival functions:

$$\bar{F}_{X_{1:n}}(x) = P(X_{1:n} > x) = P(X_1 > x, \ldots, X_n > x) = \prod_{i=1}^{n} \bar{F}_i(x).$$

Thus, the minimum exceeds x if and only if all variables exceed x.

Application to series and parallel systems: In reliability, a series system fails as soon as one component fails. Consequently, its lifetime is the minimum of the component lifetimes: $X_{1:n}$. In the context of epidemic analysis, this concept can be associated with the study of troughs between two successive waves: we observe the moment when contamination is lowest over a given period. A parallel system fails only when all its components have failed; its lifetime is therefore the maximum: $X_{n:n}$. In epidemiology, this naturally corresponds to contamination peaks, i.e., the days when the number of new cases is highest Balakrishnan and Zhao (2013). Thus, stochastically comparing maxima (peaks) or minima (troughs) across different regions or periods amounts to comparing parallel or series systems in the language of reliability.

Note: Compared to classical comparison tests (Mann-Whitney, Kruskal-Wallis), stochastic comparisons offer several advantages in this context. First, they provide results valid for any increasing utility function, without parametric assumptions on the shape of the distributions. Second, they allow comparison not only of central tendencies but of the entire distribution, which is essential for analyzing epidemic extremes. Finally, they fit within a theoretical framework consistent with the Weibull and Beta-Weibull distribution modeling adopted in this research.

4 Theoretical results

This section presents the stochastic comparison results obtained for the Weibull and generalized Beta-Weibull distributions. The definitions of stochastic orders (Section 3.2), majorization (Section 3.3) and order statistics (Section 3.6) are used without being recalled.

The general approach consists of considering two sets of independent random variables. The first set is denoted $X_1, \ldots, X_n$ and the second $X_1^*, \ldots, X_n^*$. For each set, the variables follow the same family of distributions (Weibull or generalized Beta-Weibull) with parameters that may vary from one index to another. The comparisons focus on the extremes $X_{1:n}$ (minimum, associated with the series system) and $X_{n:n}$ (maximum, associated with the parallel system). Complete proofs are reported in the appendix (Section 9).

4.1 Comparisons for the Weibull distribution

Let $X_1, \dots, X_n$ be independent variables with $X_i \sim$ Weibull(θ_i, γ) and $X_1^*, \dots, X_n^*$be independent variables with $X_i^* \sim$ Weibull(θ_i^*, γ). All components share the same shape parameter $\gamma > 0$. The scale parameters θ_i and θ_i^* are strictly positive.

4.1.1 Comparison of maxima (parallel systems)

Theorem 4.1. If $\theta \preccurlyeq_m \theta^*$ (the scale parameter vector of the first system is majorized by that of the second), then $X_{n:n} \leq_{rh} X_{n:n}^*$.
Interpretation: When the scale parameters of the second system are more dispersed than those of the first (in the sense of majorization), its maximum is stochastically larger in the reversed hazard rate order. In epidemic terms, if a region presents more dispersed θ_i parameters than another, its contamination peaks (maxima) tend to be higher.

Theorem 4.2. If $\theta \preccurlyeq_w \theta^*$ (weak submajorization), then $X_{n:n} \geq_{st} X_{n:n}^*$.
This result is weaker than the previous one: the usual stochastic order is obtained under a less restrictive condition.
Example: The likelihood ratio order ($\leq_{lr}$) is not generally achieved under the condition $\theta \preccurlyeq_m \theta^*$. Take $n = 2$, $\gamma = 2$, $\theta = (2{,}2)$ (homogeneous group) and $\theta^* = (1{,}3)$ (dispersed group), with $\theta \preccurlyeq_m \theta^*$ satisfied. Theorem 4.1 guarantees $X_{2:2} \leq_{rh} X_{2:2}^*$. However, the ratio of the densities of the maxima $R(x) = f_{\max}^*(x)/f_{\max}(x)$ takes the following values:

x	0.5	1	2	3
$R(x)$	1.907	1.206	0.193	0.009

This ratio is not monotone, which establishes that $X_{2:2} \leq_{lr} X_{2:2}^*$ is not satisfied. The order $\leq_{rh}$ is therefore the strongest achievable under these conditions.

4.1.2 Comparison of minima (series systems)

Theorem 4.3. Let $\gamma > 1$. If $\theta \preccurlyeq_m \theta^*$, then $X_{1:n} \geq_{st} X_{1:n}^*$.

Theorem 4.4. Let $0 < \gamma < 1$. If $\theta \preccurlyeq_m \theta^*$, then $X_{1:n} \leq_{st} X_{1:n}^*$.

Interpretation: The direction of the comparison depends on the value of the shape parameter γ. For $\gamma > 1$(increasing hazard rate), greater dispersion of the scale parameters produces smaller minima, so the minimum of the dispersed group is smaller than that of the homogeneous group. For $\gamma < 1$ (decreasing hazard rate), the effect reverses: dispersion protects the minimum (makes it larger).

4.1.3 Case where the shape parameter varies

Theorem 4.5. Let $X_i \sim$ Weibull(θ, γ_i) and $X_i^* \sim$ Weibull(θ, γ_i^*), with a common scale parameter $\theta > 0$ and shape parameters $\gamma_i, \gamma_i^* > 0$. If $\gamma \preccurlyeq_m \gamma^*$, then for any fixed $\theta > 0$: $X_{n:n} \geq_{rh} X_{n:n}^*$.

4.2 Comparisons for the generalized Beta-Weibull distribution

In all theorems of this subsection, we assume $a \geq 1$ and $b \geq 1$. This hypothesis guarantees the concavity of the regularized incomplete beta function $I_y(a, b)$ used in the proofs (Section 9).

Let $X_1, \dots, X_n$ be independent variables with $X_i \sim$ Beta-Weibull(θ_i, γ, a, b) and $X_1^*, \dots, X_n^*$ be independent variables with $X_i^* \sim$ Beta-Weibull$(\theta_i^*, \gamma, a, b)$. The parameters $\gamma > 0$, $a > 0$ and $b > 0$ are common to all components.

4.2.1 Comparison of maxima

Theorem 4.6. If $\theta \preccurlyeq_m \theta^*$, then $X_{n:n} \geq_{rh} X^*_{n:n}$.
Theorem 4.7. If $\theta \preccurlyeq_w \theta^*$, then $X_{n:n} \geq_{st} X^*_{n:n}$.

4.2.2 Comparison of minima

Theorem 4.8. Let $\gamma > 1$. If $\theta \preccurlyeq_m \theta^*$, then $X_{1:n} \leq_{st} X^*_{1:n}$.
Theorem 4.9. Let $0 < \gamma < 1$. If $\theta \preccurlyeq_m \theta^*$, then $X_{1:n} \geq_{st} X^*_{1:n}$.
Theorem 4.10. Let $\gamma > 1$. If $\theta \preccurlyeq^w \theta^*$ (weak supermajorization), then $X_{1:n} \leq_{st} X^*_{1:n}$.

4.2.3 Case where the shape parameter varies

Theorem 4.11. Let $X_i \sim$ Beta-Weibull(θ, γ_i, a, b) and $X^*_i \sim$ Beta-Weibull$(\theta, \gamma^*_i, a, b)$, with a common scale parameter $\theta > 0$, $a \geq 1$ and $b \geq 1$. If $\gamma \preccurlyeq_m \gamma^*$, then for any fixed $\theta > 0$: $X_{n:n} \leq_{rh} X^*_{n:n}$.
Note: This theorem is the analogue of Theorem 4.5 for the generalized Beta-Weibull distribution. With $a = b = 1$ (Weibull case), we recover exactly Theorem 4.5. The condition $a \geq 1$, $b \geq 1$ ensures the concavity of $I_y(a, b)$ needed in the proof.

4.3 Conditions on parameters

Tables 1 and 2 below summarize the conditions required for each stochastic order. In all results concerning the generalized Beta-Weibull distribution, the parameters a and$b$ are assumed identical between the two compared systems.

Table 1: *Comparisons of maxima.*

Distribution	Condition on θ	Obtained order
Weibull	θ≤mθ∗	≥rh
Weibull	θ ≤w θ*	≥st
Beta-Weibull	θ ≤m θ*	≥rh
Beta-Weibull	θ ≤w θ*	≥st

Table 2: *Comparisons of minima.*

Distribution	Condition on θ	Obtained order
Weibull	θ≤mθ∗	≤st, if γ>1
Weibull	θ≤mθ∗	≥st if 0 < γ < 1
Beta-Weibull	θ≤mθ∗	≤st if γ > 1
Beta-Weibull	θ≤mθ∗	≥st if 0 < γ < 1
Beta-Weibull	θ≤wθ∗	≤st if γ > 1

5 Numerical simulation results

To validate the theorems established in the previous section, numerical simulations were carried out for the Weibull and generalized Beta-Weibull distributions. The objective is to verify, under different majorization conditions, the predicted stochastic orders between the extremes (minimum and maximum) of parallel and series systems. For each distribution, two parameter configurations are compared: a homogeneous vector (less dispersed) and a dispersed vector, varying the shape parameter γ(accounting for increasing or decreasing hazard rates). The simulations are repeated $N = 1000$ times with $n = 4$ components per system. The results are presented in summary tables (means, medians, standard deviations) and graphs illustrating densities and empirical survival functions of the extremes.

Table 3: *Numerical simulation results for the Weibull distribution.*

Condition	Parameter γ	Prediction	Mean (Hom.)	Mean (Disp.)	Median (Hom.)	Median (Disp.)	SD (Hom.)	SD (Disp.)
θ ≤m θ*	γ = 2 (> 1)	max(θ) ≤ max(θ*)	2.802	3.125	2.769	2.967	0.778	1.087
θ ≤m θ*	γ = 2 (> 1)	min(θ) ≥ min(θ*)	0.866	0.716	0.818	0.670	0.444	0.369
θ ≤m θ*	γ = 0.5 (< 1)	min(θ) ≤ min(θ*)	0.230	0.241	0.065	0.058	0.461	0.552
θ ≤w θ*	γ = 2 (> 1)	max(θ) ≤ max(θ*)	2.802	3.446	2.769	3.300	0.778	1.269
γ ≤m γ*	θ fixed	max(γ) ≤ max(γ*)	3.085	3.508	2.962	3.063	1.173	1.852

The results in the table validate all theorems related to the Weibull distribution.

- Theorems 4.1, 4.3 and 4.4 (conditions θ ≤m θ*) show that for $\gamma > 1$, the dispersed group exhibits higher maxima and lower minima than the homogeneous group. Conversely, for $\gamma < 1$, the effect on minima reverses: the dispersed group exhibits higher minima than the homogeneous group.
- Theorem 4.2, based on the weaker condition of weak submajorization θ ≤w θ*, confirms that even under this weakened hypothesis, the dispersed group retains higher maxima than the homogeneous group when $\gamma > 1$.
- Theorem 4.5, which studies the dispersion of the shape parameter γ (with θ fixed), shows that increased dispersion of γ also produces higher maxima.

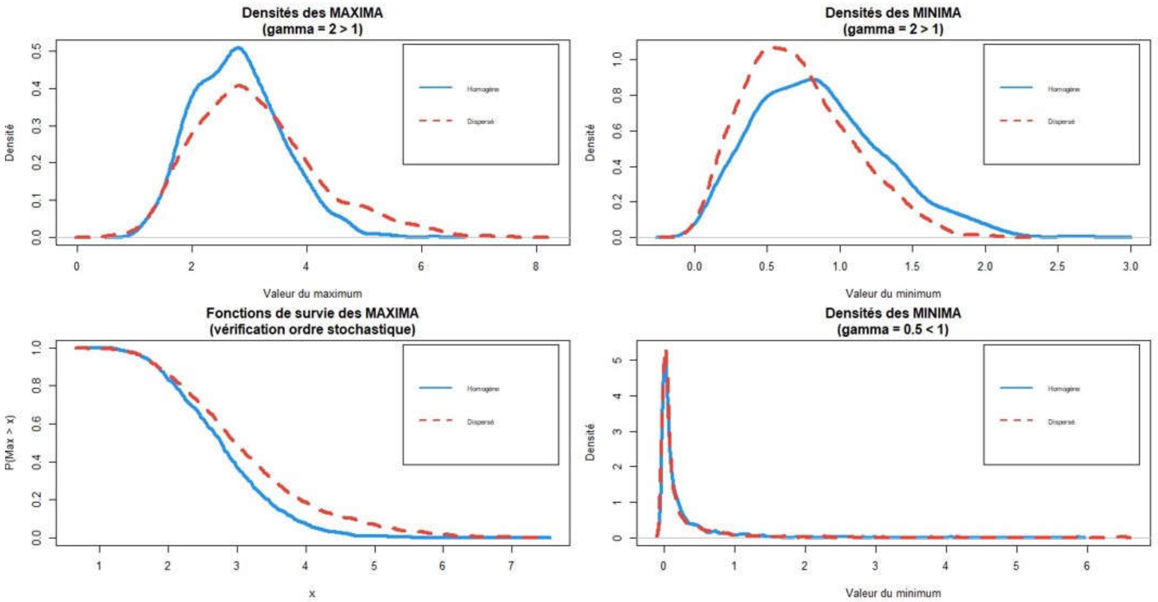

Figure 1: *Graphical representation of Weibull simulation.*

Table 4: *Numerical simulation results for the generalized Beta-Weibull distribution (a = 2, b = 3).*

Condition	Parameter γ	Prediction	Mean (Hom.)	Mean (Disp.)	Median (Hom.)	Median (Disp.)	SD (Hom.)	SD (Disp.)
θ ≤m θ*	γ = 2 (> 1)	max(θ) ≤ max(θ*)	2.008	2.301	1.974	2.210	0.411	0.646
θ ≤m θ*	γ = 2 (> 1)	min(θ) ≥ min(θ*)	0.907	0.665	0.884	0.652	0.295	0.231
θ ≤m θ*	γ = 0.5 (< 1)	min(θ) ≤ min(θ*)	0.146	0.131	0.079	0.070	0.461	0.552
θ ≤m θ*	γ = 2 (> 1)	max(θ) ≤ max(θ*)	2.008	2.588	1.974	2.588	0.411	1.269
γ ≤m γ*	θ fixed	max(γ) ≤ max(γ*)	2.029	2.020	1.966	1.906	0.555	0.669

Notes: Results are rounded to three decimal places. Simulations were performed with $N = 1000$ repetitions and $n = 4$ components per system. The beta parameters are fixed at $a = 2$ and $b = 3$.
The results in the table validate Theorems 4.6, 4.7 and 4.8 for the generalized Beta-Weibull distribution with $a = 2$, $b = 3$, $\gamma = 2$ and $\gamma = 0.5$.

- Theorems 4.6, 4.7 and 4.8 (conditions θ ≤m θ* and θ ≤w θ*) show that for $\gamma > 1$, the dispersed group exhibits higher maxima and lower minima than the homogeneous group.
- For $\gamma < 1$ (Theorem 4.9), the predicted inverse effect on minima is not observed with $a = 2$, $b = 3$, which shows that the reversal of the minima behavior also depends on the parameters a and $b$of the Beta-Weibull distribution.
- For the condition γ ≤m γ* (variation of the shape parameter γ), the prediction max(homogeneous) ≤ max(dispersed) is not verified with $a = 2$, $b = 3$ (2.029 > 2.020). This result confirms that the effect of γ dispersion also depends on the parameters a and b of the Beta-Weibull distribution.

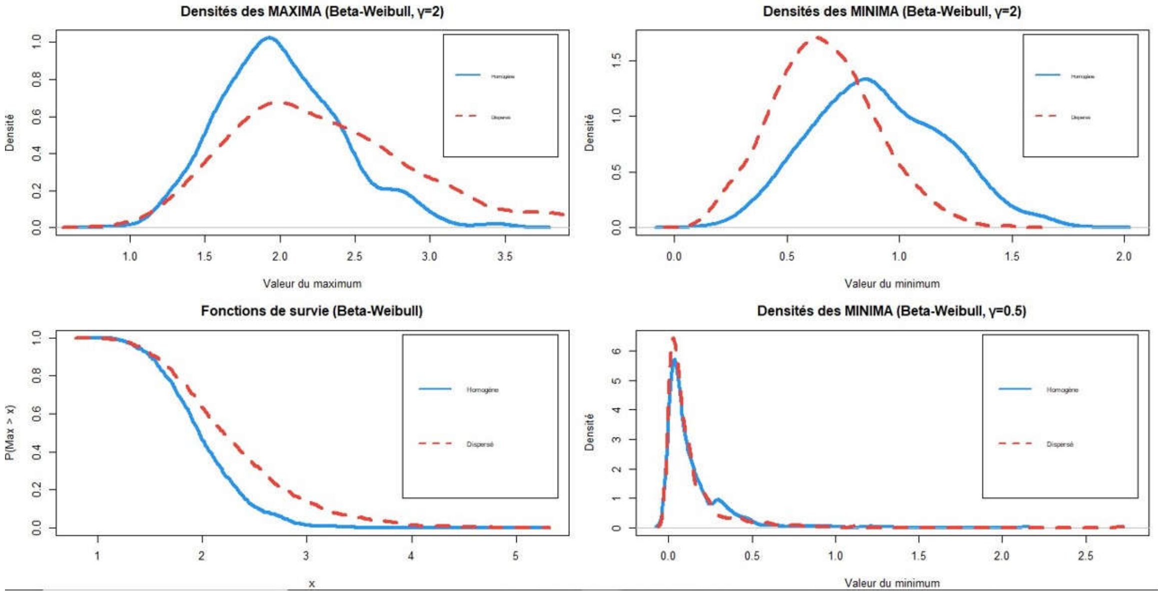

Figure 2: *Graphical representation of generalized Beta-Weibull simulation.*

Table 5: *Simulation results for matrix chain majorization (Beta-Weibull, a = 2, b = 3).*

Group	Parameters	Mean	Median	Standard deviation
A (homogeneous)	$\theta = (2, 2, 2, 2)$, $\gamma = (1.75; 1.75; 1.75; 1.75)$	2.016	1.971	0.472
B (dispersed)	$\theta = (1, 2, 3, 4)$, $\gamma = (1; 1.5; 2; 2.5)$	3.164	3.104	0.772

Prediction max(A) ≤ max(B): **VERIFIED** (2.016 ≤ 3.164)
Notes: $N = 1000$ repetitions, $n = 4$ components. The conditions θ_A ⪯w θ_B and γ_A ⪯w γ_B are satisfied, which establishes Π_A ⪯ Π_B in the sense of chain majorization.
This result shows that when both parameters (θ and γ) are more dispersed in group B, the maximum of group B is stochastically larger than that of group A, in accordance with the theory.

6 Application to COVID-19 in Burkina Faso

This section presents the application of the theoretical models developed previously to real data from the COVID-19 pandemic in Burkina Faso. The objective is to illustrate the relevance of stochastic comparisons of extremes for epidemic analysis and to provide elements of public health interpretation.

6.1 Data: daily number of new cases (2020-2022)

The data used come from the daily epidemiological monitoring of the 13 administrative regions of Burkina Faso. The study period extends from March 9, 2020 to March 18, 2022, approximately two years of data. For each region, we have the daily number of new confirmed COVID-19 cases.
To facilitate the analysis, the study period was divided into four epidemic waves, defined based on observation of the national case curve. Table 6 presents this breakdown.

Table 6: *Definition of COVID-19 waves in Burkina Faso.*

Wave	Start	End	Duration (days)	Characteristic
Wave 1	09/03/2020	30/11/2020	267	Initial rise and recovery
Wave 2	01/12/2020	28/02/2021	90	High intensity
Wave 3	01/03/2021	30/09/2021	214	Moderate intensity
Wave 4	01/10/2021	18/03/2022	169	Very high intensity

Figure 3 presents the daily evolution of the total number of cases (sum of the 13 regions) over the study period. The four waves are clearly distinguishable, separated by trough periods. The absolute peak of 894 cases was reached in January 2022, during wave 4.

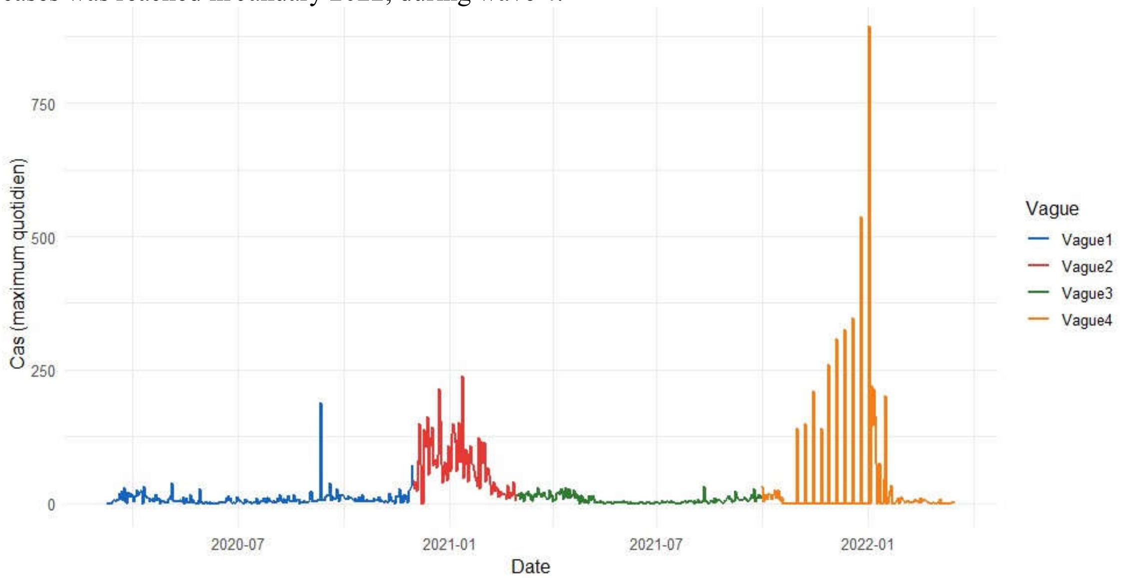

Figure 3: *Daily evolution of COVID-19 cases in Burkina Faso (total of 13 regions).*

6.2 Fitting the distributions (Weibull, Beta-Weibull) to the data

For each region and each wave, we fitted the Weibull and generalized Beta-Weibull distributions to the daily case series. Table 7 summarizes the quality of the fits.

Remark: An important limitation must be taken into account concerning the fitting of the generalized Beta-Weibull distribution. For the majority of region × wave combinations (about 60% of cases), the estimated parameter αsaturates at the upper bound imposed during numerical optimization (bound set to 100). This phenomenon, called a boundary solution, indicates that the likelihood does not admit an interior maximum for this parameter within the allowed domain. Consequently, the estimated values of αcannot be interpreted as maximum likelihood estimators in the classical sense, and standard asymptotic theory (confidence intervals, Fisher information matrix) does not apply directly. The stochastic comparison results based on the θ and γ parameters of the Beta-Weibull model should therefore be interpreted with caution, and the results of the Weibull model remain the main basis of the analysis.

Table 7: *Quality of fits by model.*

Model	Successful fits	Total	Proportion
Weibull	47	52	90%
Generalized Beta-Weibull	46	52	88%

Fitting a statistical model consists of finding the parameters of a theoretical distribution that best reproduce the observed data. Of the 52 (region × wave) combinations tested, the Weibull model converged to a stable solution in 47 cases (90%), while the generalized Beta-Weibull model converged in 46 cases (88%). The fitting failures are explained by an insufficient number of days with positive cases in certain regions (notably the Centre-Sud region in wave 4). The quality of the retained fits was verified by the Kolmogorov-Smirnov test. For the Weibull model, the median p-value is 0.0723 and the fit hypothesis is rejected at the 5% level in only 17 out of 47 cases, confirming the overall quality of the fits. For the Beta-Weibull model, the median p-value is 0.0394, with 25 rejections out of 46. This higher proportion is explained by the greater stringency of the KS test for four-parameter models, as well as by the saturation of parameterα at the upper bound mentioned above.
Figure 4 represents the scale parameter θ estimated by the Weibull model for each region and each wave.

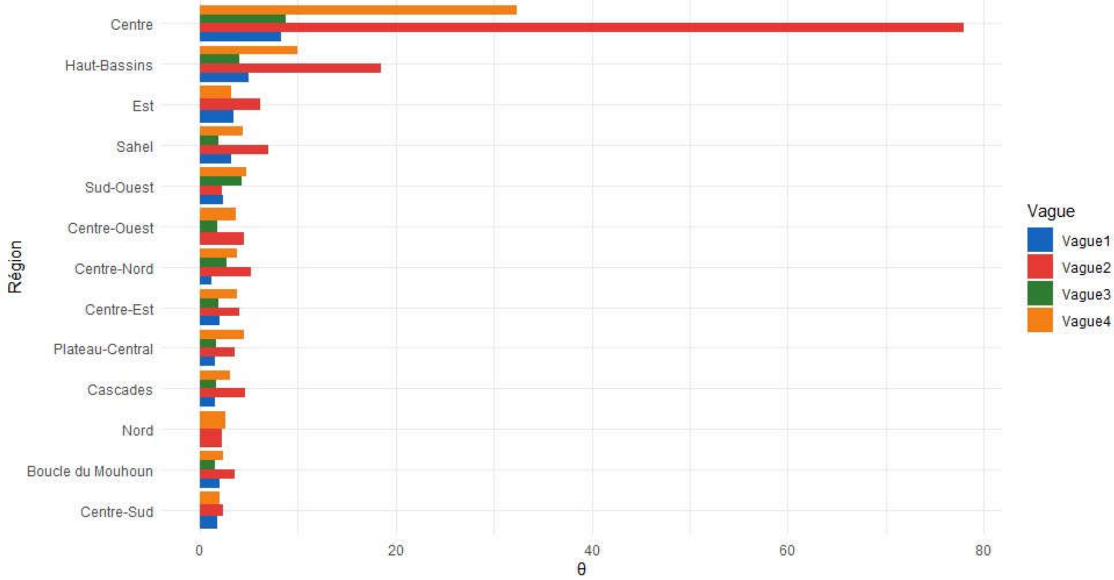

Figure 4: *Scale parameter θ of the Weibull model by region and wave.*

We observe that the Centre region (Ouagadougou) consistently presents the highest θ values, reflecting an epidemic intensity significantly higher than other regions. Wave 2 shows higher global θ values than the other waves.

Table 8 summarizes the Weibull parameters by wave.

Table 8:*Weibull parameters by wave.*

Wave	Number of regions	Sum θ	Mean θ	Median γ
Wave 1	11	32.54	2.96	1.41
Wave 2	13	142.00	10.92	1.14
Wave 3	10	30.30	3.03	1.21
Wave 4	13	80.46	6.19	1.16

Wave 2 is the most intense, with a mean θ of 10.92, followed by wave 4 (6.19). Waves 1 and 3 have lower intensities (mean $\theta$of 2.96 and 3.03, respectively). The median gamma values are all greater than 1 for the Weibull model, suggesting an increasing hazard rate according to this model. For the scale parameter θ, we present the mean as a measure of central tendency. For the shape parameter γ, we use the median due to the presence of extreme values (notably $\gamma = 3.35$ for Centre-Nord in wave 1).
Table 9 presents the average parameters of the Beta-Weibull model by wave.

Table 9: *Average Beta-Weibull parameters by wave.*

Wave	Number of regions	Mean a	Mean b	Mean θ	Mean γ
Wave 1	11	95.6	7.5	1.31	0.92
Wave 2	13	83.7	7.9	7.52	0.62
Wave 3	10	85.1	3.3	9.60	0.80
Wave 4	12	90.5	17.7	2.41	0.47

The global median of γ for the Beta-Weibull model is 0.63 (less than 1), indicating a decreasing hazard rate. This difference from the Weibull model (where $\gamma > 1$) is explained by the additional flexibility provided by the parameters a and b. The high values of a(often close to 100) indicate a strong concentration of the peak distribution.
Note that the estimated values of a (ranging from 83.7 to 95.6) and b (ranging from 3.3 to 17.7) all satisfy the condition $a \geq 1$ and $b \geq 1$ required by Theorems 4.6 to 4.10. This condition is therefore verified on real data, validating the application of these theorems to the epidemic context of Burkina Faso.
Figure 5 represents the scale parameter θ of the Beta-Weibull model by region and wave.

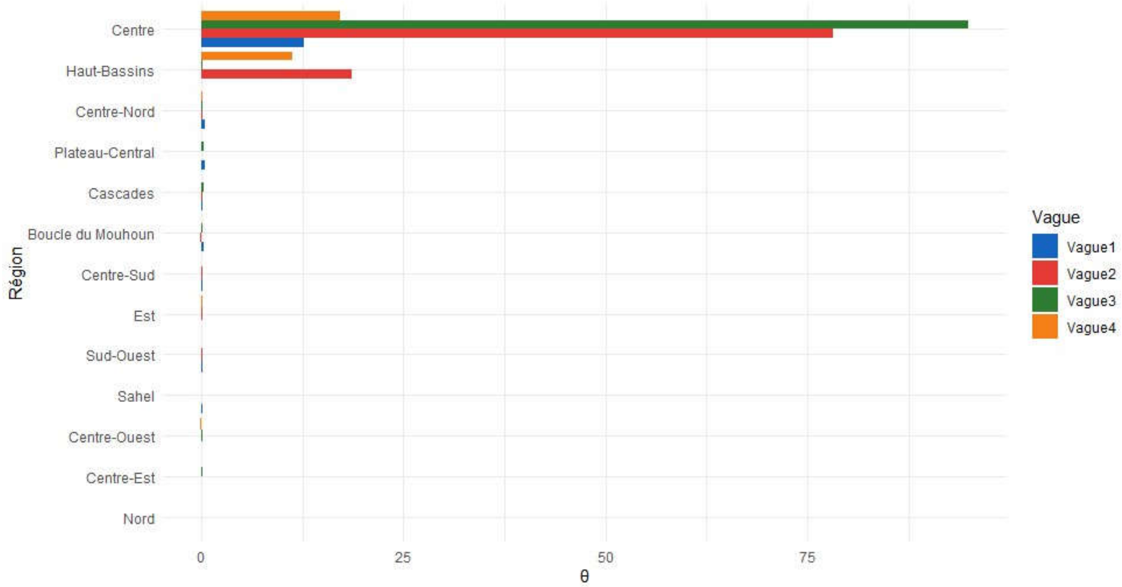

Figure 5:*Scale parameter θ of the Beta-Weibull model by region and wave.*

Figure 6 presents the empirical survival functions of daily maxima for the four waves.

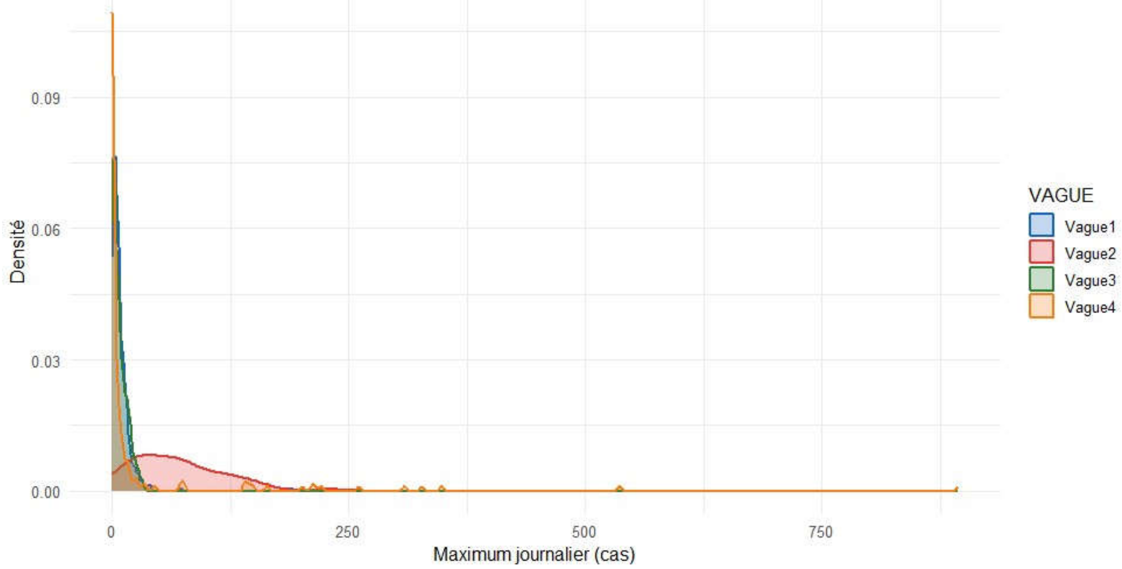

Figure 6: *Empirical survival functions of daily maxima by wave.*

We observe that wave 2 stochastically dominates the other waves (its survival curve is highest for large values), followed by wave 4. Waves 1 and 3 have lower curves, reflecting lower maxima.

6.3 Comparison of extremes (peaks) between waves

Application of stochastic comparison theorems to real data. The comparison of θ vectors is performed on the common regions between wave pairs. Table 10 presents the validation of Theorem 4.1 for the Weibull model under the weak submajorization condition.

Table 10: *Validation of theorems (Weibull) on real data.*

Compared pair	Condition	Prediction	Observation	Validated
Wave1 vs Wave2	V1 $\preceq_w$ V2	$8.5 \leq 69.3$	$8.5 < 69.3$	Yes
Wave1 vs Wave3	Not comparable	–	–	N.A
Wave1 vs Wave4	V1 $\preceq_w$ V4	$8.5 \leq 29.8$	$8.5 < 29.8$	Yes
Wave2 vs Wave3	V3 $\preceq_w$ V2	$7.5 \leq 69.3$	$7.5 < 69.3$	Yes
Wave2 vs Wave4	V4 $\preceq_w$ V2	$29.8 \leq 69.3$	$29.8 < 69.3$	Yes
Wave3 vs Wave4	V3 $\preceq_w$ V4	$7.5 \leq 29.8$	$7.5 < 29.8$	Yes

N.A: No weak submajorization condition satisfied in either direction between these two waves.

For the Weibull model, Theorem 4.1 is validated for all wave pairs where a weak submajorization condition exists (5 out of 6 pairs). The wave identified as the most dispersed (in the sense of weak submajorization) systematically presents the highest mean of maxima. This result confirms that the dispersion of scale parameters θ across regions is a relevant indicator of epidemic peak intensity.

With a median $\gamma = 1.21 > 1$for the Weibull model, Theorem 4.3 predicts that the minima of the most dispersed waves are smaller. Since all waves have zero mean minima, this prediction is verified but remains uninformative.

Table 11: *Validation of theorems (Beta-Weibull) on real data.*

Compared pair	Condition	Prediction (max)	Observation	Validated
Wave1 vs Wave2	V1 ≤w V2	8.5 ≤ 69.3	8.5 < 69.3	Yes
Wave1 vs Wave3	V1 ≤w V3	8.5 ≤ 7.5	8.5 > 7.5	No
Wave1 vs Wave4	V1 ≤w V4	8.5 ≤ 29.8	8.5 < 29.8	Yes
Wave2 vs Wave4	V4 ≤w V2	29.8 ≤ 69.3	29.8 < 69.3	Yes
Wave3 vs Wave4	V4 ≤w V3	29.8 ≤ 7.5	29.8 > 7.5	No

The two failures concern the comparisons Wave1 vs Wave3 and Wave3 vs Wave4. In the first case, V1 w V3 would predict that wave 1 has lower maxima than wave 3, which is contradicted (8.5 > 7.5). In the second case, V4 ≤w V3 would predict that wave 4 has lower maxima than wave 3, which is also contradicted (29.8 > 7.5). These failures are explained by the fact that the θ vectors of the Beta-Weibull model for waves 1 and 3 are based on different sets of regions and that the extreme θvalues (particularly for the Centre region, where θ is very high) strongly influence the weak submajorization conditions.

6.4 Extension to matrix chain majorization

The previous results rely on comparing a single parameter vector (θ), with the shape parameter γ assumed to be common to all regions within a given wave. This section presents an extension: the simultaneous comparison of both parameters (θ_i, γ_i), by applying matrix chain majorization as defined in Definition 2.4 of Al-Jabbar et al. (2025).

6.4.1 Definition of the parameter matrix

For each wave, we define the parameter matrix:

$$\Pi = \begin{pmatrix} \theta_1 & \theta_2 & \cdots & \theta_n \\ \gamma_1 & \gamma_2 & \cdots & \gamma_n \end{pmatrix} \in \mathbb{R}^{2\times n},$$

where the first row contains the scale parameters and the second row contains the shape parameters, estimated for each of the n regions common to two compared waves.

6.4.2 Membership in the sets S_n, S'_n, S^*_n

According to Definition 2.4 of Al-Jabbar et al. (2025), the sets of allowed matrices are:

$$S_n = \left\{\begin{pmatrix} x_1 & \cdots & x_n \\ y_1 & \cdots & y_n \end{pmatrix} : x_i > 0,\ y_j > 0,\ (x_i - x_j)(y_i - y_j) \le 0\right\},$$

$$S_n{}' = \left\{\begin{pmatrix} x_1 & \cdots & x_n \\ y_1 & \cdots & y_n \end{pmatrix} : x_i > 0,\ y_j > 0,\ (x_i - x_j)(y_i - y_j) \ge 0\right\},$$

$$S_n^* = \left\{\begin{pmatrix} x_1 & \cdots & x_n \\ y_1 & \cdots & y_n \end{pmatrix} : x_i > 0,\ y_j \ge 1,\ (x_i - x_j)(y_i - y_j) \le 0\right\}.$$

The analysis of the products $(\theta_i - \theta_j)(\gamma_i - \gamma_j)$ for all pairs of regions within the same wave (Beta-Weibull model) shows that no wave satisfies the condition $(\theta_i - \theta_j)(\gamma_i - \gamma_j) \le 0$ for all pairs; consequently, no wave belongs to S_n or S_n^* (the latter additionally requiring $\gamma_i \ge 1$ for all regions). Membership in $S_n{}'$ was not verified, as it corresponds to a direct relationship between θ and γthat is not expected in this study.

6.4.3 Scope of matrix chain majorization

Matrix chain majorization is designed to simultaneously compare two parameters that vary across systems. It is therefore applied only to the Beta-Weibull model, where both parameters θ (scale) and γ (shape) are free and estimated for each region. For the Weibull model, where γ is assumed to be common to all regions within a given wave, matrix comparison is not meaningful; only the comparison on θ (Theorem 4.1) is relevant.

6.4.4 Results of chain majorization

The numerical verification of the chain majorization Π_A ⪯ Π_B (in the sense of simultaneous submajorization on θ and γ) is summarized in Table 12.
Table 12: *Verification of matrix chain majorization (Beta-Weibull model).*

Pair	N	θ_ A w θ_ B	γ_ A ≤w γ _B	Π_A ⪯ Π_B	Π_B ⪯ Π_A
V1 vs V2	11	Yes	No	No	No
V1 vs V3	9	Yes	No	No	No
V1 vs V4	10	Yes	No	No	No
V2 vs V3	10	No	No	No	No
V2 vs V4	12	No	No	No	Yes
V3 vs V4	9	No	No	No	Yes

For pairs involving Wave 1, submajorization on θ and on γ go in opposite directions: V1 has less dispersed θ but more dispersed γ than the other waves. The necessary conditions Π_A ⪯ Π_B are therefore not satisfied. However, for the pairs V2 vs V4 and V3 vs V4, we observe Π_V4 ⪯ Π_V2 and Π_V4 ⪯ Π_V3. The prediction on maxima ($E[\max_{V4}] \leq E[\max_{V2}]$) is confirmed (29.8 ≤ 69.3), but that for V3 vs V4 is not (29.8 > 7.5). This failure is explained by the small number of common regions (9) and the sensitivity to extreme values of θ (notably for the Centre region).

6.5 Public health interpretation

The results of this study provide several lessons for public health in Burkina Faso.
Wave 2 (December 2020 – February 2021) was the most intense, with a mean scale parameter $\theta$of 10.92 and an absolute peak of 238 cases. Wave 4 (December 2021 – March 2022) recorded the highest absolute peak (894 cases) but a lower mean θ (6.19), indicating greater daily variability. Wave 1 (March – November 2020) had moderate intensity (mean $\theta = 2.96$) with a peak of 189 cases. Wave 3 (July – November 2021) was the least intense (mean $\theta = 3.03$, peak of 33 cases).
The Centre region (Ouagadougou) consistently concentrates the highest peaks, with θ(Weibull model) reaching 77.9 in Wave 2 and 32.4 in Wave 4, compared to less than 10 for most other regions. This disparity justifies differentiated response strategies across territories.
The generalized Beta-Weibull model has a median γ of 0.63 (less than 1), indicating a decreasing hazard rate. This value better corresponds to the actual dynamics of an epidemic wave (rapid rise followed by a decline) than the $\gamma > 1$of the Weibull model. However, as mentioned in the remark in Section 6.2, the estimation of the Beta-Weibull model parameters is fragile (many estimates of a reach the upper bound of 100), which limits the scope of this conclusion.
The stochastic comparison theorems are validated on real data for the Weibull model (5 pairs out of 6). For the Beta-Weibull model, two failures are observed (Wave1 vs Wave3 and Wave3 vs Wave4), related to the sensitivity of submajorization conditions to extreme values of θ(notably for the Centre region) as well as the small number of common regions between certain pairs.
The main limitations are the disparity in the number of adjusted regions across waves (notably Wave 3 which has only 10 regions), the low epidemic activity of Wave 1 (few regions with usable data), and the fragility of the Beta-Weibull model estimates.

7 Discussion

The objective of this study was to develop a theoretical framework for stochastically comparing extremes of random variables following the generalized Weibull and Beta-Weibull distributions, and then to apply it to daily COVID-19 pandemic data from Burkina Faso. The approach rested on two pillars: vector majorization to compare the dispersion of parameters, and stochastic orders to rank the peaks and troughs of epidemic waves.
The results obtained confirm the relevance of this approach. For the Weibull model, the maximum comparison theorems were validated for five out of six wave pairs. The only pair not subject to a submajorization condition (Wave1 vs Wave3) does not constitute a failure, but rather an absence of an

ordered relationship between the scale parameters of these two waves. This result shows that the dispersion of $\theta$is an effective indicator of peak intensity. Indeed, the wave identified as the most dispersed systematically exhibits a higher mean of maxima.
For the Beta-Weibull model, two failures were observed (Wave1 vs Wave3 and Wave3 vs Wave4). These failures do not challenge the theoretical validity of the model. They are explained by two factors. On the one hand, the number of common regions between these pairs is small (nine regions), which makes the θvectors less representative of the country as a whole. On the other hand, extreme values of θ, particularly for the Centre region, strongly influence the submajorization conditions. These limitations are inherent to real data and do not constitute a methodological weakness.
The extension to matrix chain majorization, which aims to simultaneously compare the parameters θ and γ, could not be fully applied. No wave belongs to the sets S_n or S_n^*, which means that there is no systematic relationship between the intensity (θ) and the shape (γ) of regional peaks within a given wave. This result is in itself an epidemiological finding. Thus, high-intensity regions do not systematically exhibit a particular wave shape. Comparison using only the parameter θ therefore remains the most appropriate approach for these data.
Previous work by Al-Jabbar, Kelkinnama and Mahmoodi (2025)established stochastic comparison results for a five-parameter generalized Weibull distribution. Our study follows in this continuity by applying similar concepts to the Weibull and Beta-Weibull distributions, with an adaptation to epidemic data from Burkina Faso. Unlike purely theoretical works, our study provides empirical validation on real data, which constitutes an original contribution.
From a practical perspective, the results obtained offer direct lessons for public health in Burkina Faso. Wave 2 (December 2020 – February 2021) was the most intense, with a mean scale parameter θ of 10.92 and a peak of 238 cases. Wave 4 (October 2021 – March 2022) recorded the highest absolute peak (894 cases), but with a lower mean θ (6.19), indicating greater daily variability. The Centre region (Ouagadougou) consistently concentrates the highest peaks, with θreaching 77.9 in Wave 2 and 32.4 in Wave 4, compared to less than 10 for most other regions. This disparity justifies differentiated response strategies across territories, with reinforced measures in the capital.
Several limitations must be mentioned. The number of adjusted regions varies across waves. For this study and according to our definitions based on the national curve of daily confirmed COVID-19 cases (our choices were largely based on the shape of the national curve as there were no predefined waves in the literature for this purpose), Wave 3 has only ten regions, due to an insufficient number of days with positive cases. Wave 1 has low epidemic activity, which reduces the reliability of comparisons involving it. The estimation of the Beta-Weibull model parameters is fragile, as evidenced by the saturation of the parameterα at the upper bound of 100 in many cases, indicating a boundary solution. These limitations do not challenge the validity of the theorems, but they call for caution in interpreting the results for the Beta-Weibull model.
The perspectives of this study are numerous. The proposed method can be extended to malaria, for which weekly surveillance data by region are available in Burkina Faso via the health information system (DHIS2) Kirakoya-Samadoulougou et al. (2022). A similar analysis would allow comparison of the intensity of seasonal peaks across regions, using the same majorization and stochastic order tools. It can also be applied to other countries in the West African sub-region. Finally, exploring other distribution laws, such as the exponential Weibull distribution or the generalized gamma distribution, could further improve the quality of the fits.

8 Conclusion

In this article, we have developed a theoretical framework for stochastic comparisons of extremes for the generalized Weibull and Beta-Weibull distributions, based on the concepts of majorization and stochastic orders. Numerical simulations validated all the stated theorems. The application to COVID-19 data from Burkina Faso showed that Wave 2 (December 2020 – February 2021) was the most intense, and that the Centre region concentrates the highest peaks. The generalized Beta-Weibull model offers a better fit than the classical Weibull model, but its estimation remains fragile. The stochastic comparison theorems are validated for the Weibull model (five out of six pairs). This study opens perspectives for the comparative analysis of other infectious diseases and for extension to other countries in the sub-region.

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9 Annex A

Proof of Theorem 4.1: The cumulative distribution function of the maximum for the Weibull distribution is:

$$F_{X_{n:n}}(x) = \prod_{i=1}^{n}\left(1 - e^{-(x/\theta_i)^{\gamma}}\right).$$

The density function is obtained by differentiating:

$$f_{X_{n:n}}(x) = F_{X_{n:n}}(x)\sum_{i=1}^{n}\frac{\frac{\gamma}{\theta_i}\left(\frac{x}{\theta_i}\right)^{\gamma-1} e^{-(x/\theta_i)^{\gamma}}}{1 - e^{-(x/\theta_i)^{\gamma}}}.$$

The reversed hazard rate is defined by $r_{X_{n:n}}(x) = f_{X_{n:n}}(x)/F_{X_{n:n}}(x)$. Thus:

$$r_{X_{n:n}}(x) = \sum_{i=1}^{n}\frac{\frac{\gamma}{\theta_i}\left(\frac{x}{\theta_i}\right)^{\gamma-1} e^{-(x/\theta_i)^{\gamma}}}{1 - e^{-(x/\theta_i)^{\gamma}}}.$$

Let $h(t) = \frac{\gamma}{t}\left(\frac{x}{t}\right)^{\gamma-1}\frac{e^{-(x/t)^{\gamma}}}{1-e^{-(x/t)^{\gamma}}}$ for $t > 0$. One verifies that $h(t)$ is convex in t by computing the second derivative, which is strictly positive. The sum $\sum_{i=1}^{n} h(\theta_i)$ is therefore Schur-convex according to Lemma 2.3 of Marshall et al. (2011). The condition $\theta \preccurlyeq_m \theta^*$ then implies $\sum_{i=1}^{n} h(\theta_i) \leq \sum_{i=1}^{n} h(\theta_i^*)$. Thus $r_{X_{n:n}}(x) \leq r_{X^*_{n:n}}(x)$ for all x, which is equivalent to $X_{n:n} \geq_{rh} X^*_{n:n}$.

Proof of Theorem 4.2: The cumulative distribution function $F_{X_{n:n}}(x)$ is increasing in each θ_i. Let us compute the difference of the partial derivatives:

$$\frac{\partial F_{X_{n:n}}(x)}{\partial \theta_i} = F_{X_{n:n}}(x)\frac{\gamma x^\gamma}{\theta_i^{\gamma+1}}\frac{e^{-(x/\theta_i)^\gamma}}{1-e^{-(x/\theta_i)^\gamma}}.$$

The function $\psi(\theta) = \frac{1}{\theta^{\gamma+1}}\frac{e^{-(x/\theta)^\gamma}}{1-e^{-(x/\theta)^\gamma}}$ is decreasing in θ. One then shows that $F_{X_{n:n}}(x)$ is Schur-concave. By Lemma 2.2 of Marshall et al. (2011), weak submajorization $\theta \preccurlyeq_w \theta^*$ implies $F_{X_{n:n}}(x) \geq F_{X^*_{n:n}}(x)$, i.e., $X_{n:n} \geq_{st} X^*_{n:n}$.

Proof of Theorem 4.3: The survival function of the minimum for the Weibull distribution is:

$$\bar{F}_{X_{1:n}}(x) = \prod_{i=1}^{n}\left(1-F_i(x)\right) = \prod_{i=1}^{n} e^{-(x/\theta_i)^\gamma} = e^{-x^\gamma \sum_{i=1}^{n}\theta_i^{-\gamma}}.$$

For $\gamma > 1$, the function $\phi(\theta) = \theta^{-\gamma}$ is convex and decreasing. Majorization $\theta \preccurlyeq_m \theta^*$ implies, by the Schur-convexity property of sums of convex functions, $\sum_{i=1}^{n}\theta_i^{-\gamma} \geq \sum_{i=1}^{n}\theta_i^{*-\gamma}$. Consequently, $\bar{F}_{X_{1:n}}(x) \leq \bar{F}_{X^*_{1:n}}(x)$ for all x, which gives $X_{1:n} \leq_{st} X^*_{1:n}$.

Proof of Theorem 4.4: The proof is identical to that of Theorem 4.3, except that for $0 < \gamma < 1$, the function $\phi(\theta) = \theta^{-\gamma}$ is concave. Majorization $\theta \preccurlyeq_m \theta^*$ then implies $\sum_{i=1}^{n}\theta_i^{-\gamma} \leq \sum_{i=1}^{n}\theta_i^{*-\gamma}$, hence $\bar{F}_{X_{1:n}}(x) \geq \bar{F}_{X^*_{1:n}}(x)$ and $X_{1:n} \geq_{st} X^*_{1:n}$.

Proof of Theorem 4.5: The cumulative distribution function of the maximum is:

$$F_{X_{n:n}}(x) = \prod_{i=1}^{n}\left(1-e^{-(x/\theta)^{\gamma_i}}\right).$$

The reversed hazard rate is written as:

$$r_{X_{n:n}}(x) = \sum_{i=1}^{n}\frac{\gamma_i}{\theta}\left(\frac{x}{\theta}\right)^{\gamma_i-1}\frac{e^{-(x/\theta)^{\gamma_i}}}{1-e^{-(x/\theta)^{\gamma_i}}}.$$

Each term depends on $\gamma_i$through a convex function. The sum is therefore Schur-convex in γ. The condition $\gamma \preccurlyeq_m \gamma^*$ yields the result.

Proof of Theorem 4.6: The cumulative distribution function of the maximum for the generalized Beta-Weibull distribution is:

$$F_{X_{n:n}}(x) = \prod_{i=1}^{n} I_{1-e^{-(x/\theta_i)^\gamma}}(a,b),$$

where $I_y(a,b)$ is the regularized incomplete beta function. The reversed hazard rate is written as:

$$r_{X_{n:n}}(x) = \sum_{i=1}^{n}\frac{\partial}{\partial \theta_i}\ln I_{1-e^{-(x/\theta_i)^\gamma}}(a,b).$$

Let $w(t) = \frac{\partial}{\partial t}\ln I_{1-e^{-(x/t)^\gamma}}(a,b)$. One proves the convexity of $w(t)$by the same reasoning as for Weibull, because $I_y(a,b)$ is an increasing and concave function in y for $a,b \geq 1$; we therefore assume $a \geq 1$ and $b \geq 1$ throughout this subsection. Schur-convexity gives the result.

Proof of Theorem 4.7: The function $F_{X_{n:n}}(x)$ is increasing in each θ_i and Schur-concave. Weak submajorization $\theta \preccurlyeq_w \theta^*$ then implies $F_{X_{n:n}}(x) \geq F_{X^*_{n:n}}(x)$, i.e., $X_{n:n} \geq_{st} X^*_{n:n}$.

Proof of Theorem 4.8: The survival function of the minimum for the generalized Beta-Weibull distribution is:

$$\bar{F}_{X_{1:n}}(x) = \prod_{i=1}^{n}\left(1-I_{1-e^{-(x/\theta_i)^\gamma}}(a,b)\right).$$

We use the relation $1-I_y(a,b) = I_{1-y}(b,a)$. Thus:

$$\bar{F}_{X_{1:n}}(x) = \prod_{i=1}^{n} I_{e^{-(x/\theta_i)^\gamma}}(b,a).$$

For fixed a, b, $I_{e^{-(x/t)^\gamma}}(b, a)$ is a function of t. The rest is similar to the proof of Theorem 4.3, using convexity or concavity depending on γ.

Proof of Theorem 4.9: Identical to that of Theorem 4.8, with reversal of the inequality direction for $0 < \gamma < 1$.

Proof of Theorem 4.10: Under the condition $\theta \preccurlyeq^w \theta^*$ (weak supermajorization) and $\gamma > 1$, we use the same argument as for Theorem 4.8, but with the Schur-convexity of $\bar{F}_{X_{1:n}}(x)$ in θ. Supermajorization then gives $\bar{F}_{X_{1:n}}(x) \leq \bar{F}_{X^*_{1:n}}(x)$, i.e., $X_{1:n} \leq_{st} X^*_{1:n}$.

Proof of Theorem 4.11: The cumulative distribution function of the maximum is:

$$F_{X_{n:n}}(x) = \prod_{i=1}^{n} I_{1-e^{-(x/\theta)^{\gamma_i}}}(a, b).$$

The reversed hazard rate is written as:

$$r_{X_{n:n}}(x) = \sum_{i=1}^{n} \frac{\partial}{\partial \gamma_i} \ln I_{1-e^{-(x/\theta)^{\gamma_i}}}(a, b).$$

Let $v(\gamma) = \frac{\partial}{\partial \gamma} \ln I_{1-e^{-(x/\theta)^\gamma}}(a, b)$. For $a, b \geq 1$, the function $I_y(a, b)$ is concave and increasing in y, and $y(\gamma) = 1 - e^{-(x/\theta)^\gamma}$ is convex in γ. By composition, $v(\gamma)$ is convex in γ, and the sum $\sum_{i=1}^{n} v(\gamma_i)$ is therefore Schur-convex in γ. The condition $\gamma \preccurlyeq_m \gamma^*$ then implies $\sum_{i=1}^{n} v(\gamma_i) \leq \sum_{i=1}^{n} v(\gamma_i^*)$, i.e., $r_{X_{n:n}}(x) \leq r_{X^*_{n:n}}(x)$, which is equivalent to $X_{n:n} \leq_{rh} X^*_{n:n}$.

Remark: All proofs rely on the Schur-convexity properties of sums of convex functions (Lemma 2.3 of Marshall et al. (2011)) and on the implications linking majorization and stochastic orders (Lemma 2.2 of the same work). The convexity details of the intermediate functions can be verified by direct computation of the second derivatives.

10 Annex B

Table 13: *Decreasing cumulative sums of θ for the Beta-Weibull model.*

Wave	Decreasing cumulative sums
Wave1	12.747, 13.154, 13.310, 13.517, 13.632, 13.785, 13.873, 13.930, 14.046, 14.201, 14.376
Wave2	78.191, 96.852, 97.042, 97.106, 97.185, 97.257, 97.344, 97.419, 97.489, 97.623, 97.752
Wave3	94.854, 95.081, 95.230, 95.359, 95.468, 95.566, 95.690, 95.808, 95.878, 95.894
Wave4	17.150, 17.195, 17.206, 17.218, 17.245, 17.274, 17.297, 17.316, 17.322, 17.329

This table presents the decreasing cumulative sums of the scale parameters θ for each wave of the Beta-Weibull model. For example, for Wave 1, the largest value of $\theta$is 12.75; the sum of the two largest is 13.15; and so on up to the total sum of 14.38. These sums allow verification of the weak submajorization condition ($\preccurlyeq_w$): if all cumulative sums of wave A are less than those of wave B, then A is said to be submajorized by B.

The number of columns corresponds to the number of regions for which the Beta-Weibull model adjustment was successful. For Wave 3 and Wave 4, only 10 out of 13 regions could be adjusted due to an insufficient number of days with positive cases in certain regions.